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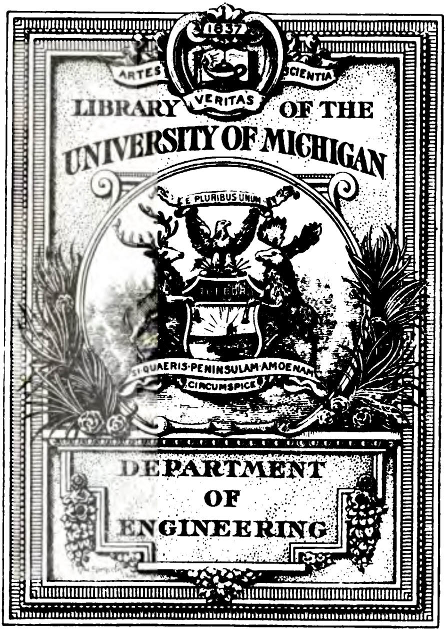

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LINKAGES:

THE

DIFFERENT FORMS AND USES OF ARTICULATED LINKS.

BY

J. D. C. DE ROOS.

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AND

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PREFACE.

The French author of this collection of examples lays no claim to originality. It is mostly gathered from English and Continental sources, but the principal indebtedness is to Professor Sylvester, whose paper, prepared for the Royal Society, was at least strongly suggestive of the applications here presented.

The subject has not even yet received the attention which is fully its due.

Some applications of the Peaucellier cell to mechanism have secured attention and approbation in a new quarter, and it seems now that a new demand may stimulate new inquiry, and lead to further valuable discovery, although the usefulness of the original cell has not yet been fully made known.

LINKAGES.

M. PEAUCELLIER, an officer of engineers of the French army, proposed in 1864, in the *Nouvelles Annales de Mathematiques*, the problem of the transformation from circular to alternating rectilinear motion, by means of articulated rods, and in 1873 he furnished the solution in the same journal.

The arrangement which he devised is extremely simple and has been applied to many problems, both in tracing curves and the mechanical solution of problems. Still it is only quite recently that this beautiful invention has received the attention that it merits, and we are largely indebted to M. Lipkin, a pupil of M. Schebicheff, of the University of St. Petersburg, who published in 1871 in the *Bulletin de St. Petersburg*, a solution of the same problem, entirely independent of the cell of Peaucellier.

Since then the rhombus of Peaucellier was made the subject of a paper by Prof. Sylvester, before the Royal Institution of Great Britain, which was published in many English journals, and was translated for the *Revue des Scientifique* of Nov. 21, 1874.

This paper, very interesting from many points of view, serves as the basis of the present essay.

We shall consider more particularly the direct practical applications of this invention, such as the tracing of conic sections and other curves, rather than its uses in mechanical calculations.

Peaucellier's apparatus consisted essentially of four equal rods or links forming an articulated rhombus, but the name is more particularly applied to the arrangement represented in Figs. 1 and 2, in which the four links are pivoted at the extremities A, B, A′ and C in such manner as to be movable in the plane of the figure, while the two opposite angles A and A′ are also united by links to the axis of rotation O.

Fig. 1.

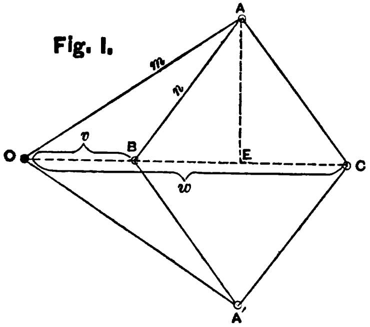
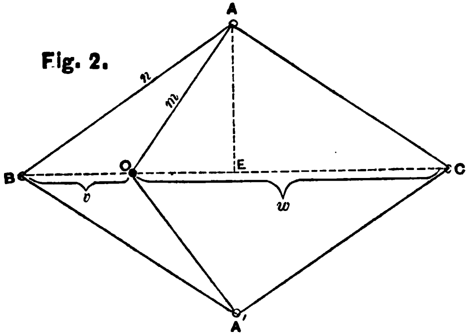


Fig. 2.



The movable joints are indicated in the figures by open circles, and the fixed joints by black ones.

The connector links OA and OA′ may

be longer or shorter than the sides of the rhombus, as shown by Figs. 1 and 2, and upon this proportion depends the position of O, whether it falls outside or inside.

The system then, is variable at will, so far as the distances OB and OC are concerned, within determinate limits.

Whatever dimensions are assumed for one of these distances, it is evident that the three points O, B and C will fall in a right line, and that the two arms OB and OC depend upon each other.

The remarkable properties of Peaucellier's cell depend upon the simple relations existing between these arms or links.

From the point A (Figs. 1 and 2) drop the perpendicular AE upon BC, and for the case in which the arms have the same direction as in Fig. 1, $OB = + v$, and $OC = + w$; while, for the case represented in Fig. 2, in which the arms have opposite directions, we make $OB = - v$ and $OC = + w$.

Let m represent the length of OA or

OA′, and n the length of each of the sides of the rhombus; we have, when the point O is on the outside:

$$\mathrm{BE} = \tfrac{1}{2}\mathrm{BC} = \tfrac{1}{2}(w - v)$$

$$m^2 = \overline{\mathrm{AE}}^2 + [v + \tfrac{1}{2}(w - v)]^2$$

and $$n^2 = \overline{\mathrm{AE}}^2 + \tfrac{1}{4}(w - v)^2.$$

When o is within the rhombus

$$\mathrm{BE} = \tfrac{1}{2}\mathrm{BC} = \tfrac{1}{2}(v + w)$$

$$m^2 = \overline{\mathrm{AE}}^2 + [\tfrac{1}{2}(v + w) - v]^2$$

and $$n^2 = \overline{\mathrm{AE}}^2 + \tfrac{1}{4}(v + w)^2$$

These equations give for the first case;

$$m^2 - n^2 = vw,$$

and for the second,

$$m^2 - n^2 = -vw;$$

so that in whatever manner we move the systems, the product of the arms is always constant.

The difference $m^2 - n^2$ is called the *modulus.* Perhaps it would be better to call it *power*, from its analogy to the equation of the hyperbola referred to its asymptotes, and we will represent it by C^2. The power may be positive or negative according as O is outside or inside the rhombus, or whether the arms have the same or opposite directions.

In the first case, the system is said to be *positive*, and in the second, *negative.*

For the greater facility in the calculation, we will henceforth consider the arms as positive, or, in other words, we will multiply the equation $m^2 - n^2 = - vw$ by $- 1$, which will give C^2 another signification, and make the difference $m^2 - n^2$, or $n^2 - m^2$ in all cases a positive quantity, whether m is greater or smaller than n.

The difference between the positive and the negative system will be simply that, in one case the arms will have the same, and in the other, opposite directions; and for both cases we shall have the general equation.

$$vw = C^2 \quad (1)$$

As to the limits between which v and w can vary, it is true they are in fact nearly equal, they may, notwithstanding, be extended to any limits between zero and infinity.

Designating by x the least value of v or w, and by y the greatest value, we shall have, Fig. 1, $w=x=m-n$ and, Fig. 2, $x=m-n$, while for both $y=m+n$, whence for the positive system

$$m=\tfrac{1}{2}(x+y),\ n=\tfrac{1}{2}(y-x),$$

and for the negative system these values interchanged

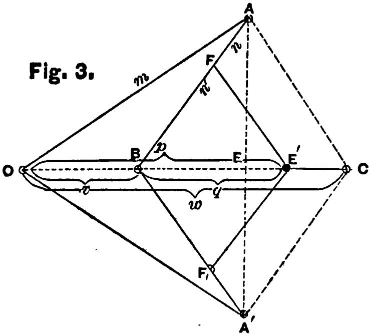
$$n=\tfrac{1}{2}(x+y),\ m=\tfrac{1}{2}(y-x).$$

In taking then m or n at convenience, x and y may take all values between zero, and infinity, and the determination of their limits has no other significance.

Before passing to the applications of these positive and negative systems, we wish to say a few words about a system, by means of which algebraic operations, of which we will treat, can often be simplified.

In Figs. 1 and 2 if we transport the sides AC and A′C, parallel to their original positions, till their intersection falls at some point E′ on the line BC (Fig. 3), then joining, as shown in the figure, the four sides to the points F and F′, there will be formed another rhombus, having for its side BF or BF′$=n'$, and if we designate by p and q the arms OE′ and BE′, we shall have:

Fig. 3.



because BE′ : BC : : BF : BA

or $q : w - v : : n' : n$

therefore $q = \frac{n'}{n}(w - v)$

and $p = v + q = v + \frac{n'}{n}(w - v)$

whence $p + q = v + 2\frac{n'}{n}(w - v)$

$$p - q = v$$

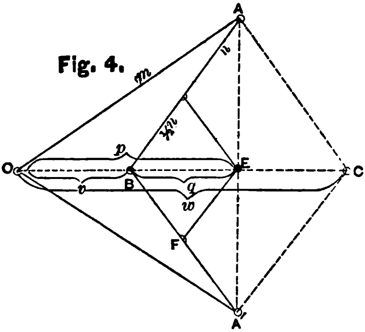
and consequently

$$p^2-q^2=v^2\left(1-2\frac{n'}{n}\right)+2\frac{n'}{n}vw=v^2\left(1-2\frac{n'}{n}\right)+2\frac{n'}{n}C^2$$

This equation is simplified by supposing, as we shall, in that which follows, that the point E′ coincides with the intersections E of the diagonals of the rhombus (Fig. 4); in other words, taking $n'=\frac{1}{2}n$. In this case the first term of the second member disappears, and there remains

$$p^2-q^2=C^2 \qquad (2)$$

Fig. 4.



In the transformed system, Fig. 5, it

is necessary to change $p^2 - q^2$ to $q^2 - p^2$, unless we wish to assign a negative value to C^2.

But the direction of p and q in relation to E may become invested when OB increases sufficiently, measuring from O (Fig. 4) or from B (Fig. 5), while the limits of p and q remain the same in both cases.

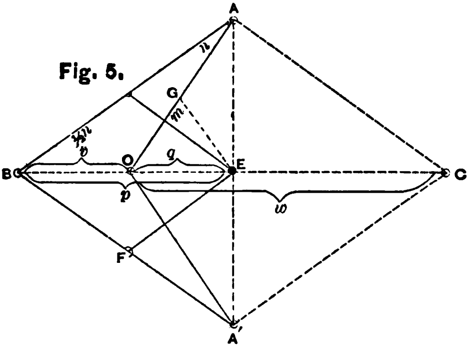


Fig. 5.

For the applications, however, it is a matter of indifference whether we employ a positive or a negative system, since in the two cases the uniting of the most salient vertices gives the same triangle, or quadrilateral, for the same limits and values of p and q.

There exists only a difference in the dimensions of the cell, and this is merely apparent since the free movement of the bars is not at all embarrassed by uniting E to the middle of the connectors.

The limits of p and q are, in virtue of Fig. 4, for the case in which we wish negative values $\sqrt{m^2-n^2}$ and m for p, o and n for q. On the other hand, in virtue of Fig. 5, the inverted values are $\sqrt{n^2-m^2}$ and n for q, o and m for p; furthermore, the values m and n can always be so taken that p and q shall remain within determinate limits.

Besides this system, of which Sylvester first pointed out the remarkable property of making the difference of the squares p^2-q^2 constant, the same author mentions another linkage, by means of which circles of long radius may be traced.

This combination is shown in Fig. 6, but theoretically it does not differ essentially from Fig. 3. In effect, from the moment that OB (Fig. 3) becomes greater than OE′, that is to say, when q becomes negative, Fig. 3 is transformed into Fig. 6, so that the equation

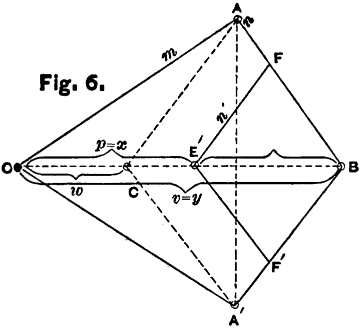


Fig. 6.

$$p^2 - q^2 = \left(1 - 2\frac{n'}{n}\right)v^2 + 2\frac{n'}{n}C^2,$$

which becomes, after substituting for q, $p - v$;

$$pv - \left(1 - \frac{n'}{n}\right)v^2 = \frac{n'}{n}C^2$$

is equally applicable to the case of Fig. 6.

We will make a special study of this form of linkage, which we will call Sylvester's linkage (losange de Sylvester), since he first gave a description of it in the *Educational Times.* We will designate by x and y respectively the distances OE′$=p$ and OB$=v$ (which have in the case the function of arms), and are measured from O (Fig. 6), so that their general properties are expressed by the equation:

$$xy-\left(1-\frac{n'}{n}\right)y^2=\frac{n'}{n}\,C^2 \qquad (3)$$

Making C^2 equal to zero, that is to say, making $m=n$, we obtain:

$$y=\frac{n}{m-n'}\,x$$

and the system is transformed into a pantograph (Fig. 7). Add further to the link E′F′ the coupling E′H in the prolongation of F′E′, the two bars CA′ and A′B can be suppressed, and the system takes the form of the common pantograph (Fig. 8).

Although the systems, which we have already considered, are comprehended in the general arrangement of Fig. 9, of which the links are in pairs of the same

Fig. 7.

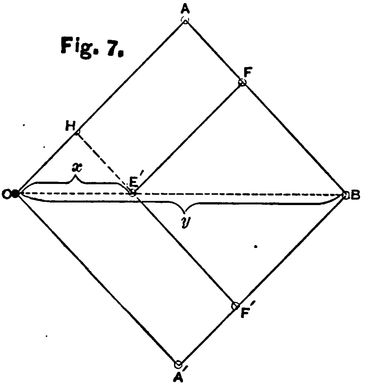
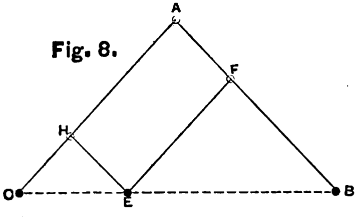


Fig. 8.



length, we have derived from preference one from the other in order to avoid, so far as possible, the change of signs.

Strictly the links should not be in pairs of equal length, but it is evident that when this equality is realized that

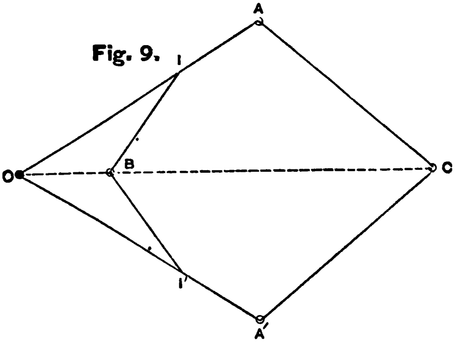


Fig. 9.

the three points O, B and C will be found in a straight line. The properties found to belong to Peaucellier's cell are independent of the length of the links. Nevertheless, we will treat from preference of symmetrical combinations.

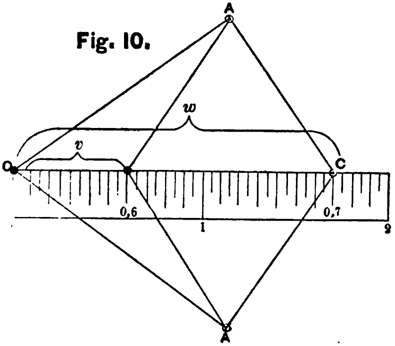
The systems considered can be coupled together and thus afford new combinations of linkages; and we give the name of *elements* to the primary systems which make up the compound system.

For the applications which we propose to study, it will suffice to employ the Peaucellier Cell. The transformed system (système varié, Fig. 5) and the linkage of Sylvester, the two latter forming, so to speak a single system.

THE PEAUCELLIER CELL AS AN INSTRUMENT FOR CALCULATIONS.

One of the most simple applications of this linkage is that which affords the reciprocal of any given ratio.

Fig. 10.



If in Fig. 10 a graduated scale OC be placed along the line of the poles BC of a positive lozenge, the point C will indicate the inverse value of OB, if the zero of the scale is placed at O, and if the power C is taken for unity of the scale.

From equation (1) we deduce

$$OB.OC=C^2$$

or $\frac{OC}{C}=\frac{1}{\frac{OB}{C}}$ in which $\frac{OC}{C}$ is the reciprocal of $\frac{OB}{C}$.

When employed thus the system is called a *reciprocator.* The lengths OA and AB (Fig. 10) being measured in millimeters $OA^2-AB^2=625$, and consequently the unit C is the square root of this number, or 25 millimeters. In the negative system the divisions of the scale would have to be continued beyond the zero point. For any other unit c the value of OC would represent $\left(\frac{C}{c}\right)^2$ times the reciprocal of OB.

Although the reciprocator is of but little practical use in finding reciprocals of given numbers, since they are determined so easily by other means, it is capable, nevertheless, of rendering good service in obtaining a series of inverse values, such, for example, as those of consecutive radii vectores of a curve.

The trace of the inverse of a curve being given, that is to say, of a curve of which the radii vectores have an inverse value, we may obtain then by the pantograph, a trace of the curve on a reduced scale.

Nearly in the same manner, and basing the operation upon the equation $OE = \sqrt{EB^2 + C^2}$, we may by means of the transformed system (Fig. 4) extract the square root of a quadratic binomial containining one variable. In this case the point E should coincide with the zero of the scale. The division of the scale, which is regarded as a unit, is a matter of indifference, provided C is measured by the same unit. The length of the connectors should be made variable, and permit giving different values to C. This system may, like the reciprocator, be employed in tracing curves.

Expanding or reducing variable quantities by the Peaucellier Cell is a somewhat more complicated operation, for it requires the joining of two of their elements. In uniting them we superpose the joints O and C as shown in Fig. 11, which comprehends a positive and a negative system. In this compound system the linkages have evidently the same liberty of motion relatively to each other as when the elements are separate; and the reciprocal relations of the distances between the centers of relation of either system, compared with the other, lead to those remarkable properties which result from the combination.

Thus in Fig. 11, having $OD.OC=C^2$ and $BO.OC=C'^2$, whence $OD=\frac{C^2}{C'^2}$ OB, we see the arm OD represents a length which is $\frac{C^2}{C'^2}$ times as great as OB, and this system becomes transformed into an ordinary pantograph by making OC nothing, and letting fall one of the links CE and CA′ upon the prolongation of the other. The same may be said of AC and CE′. It is true that the powers C^2 and

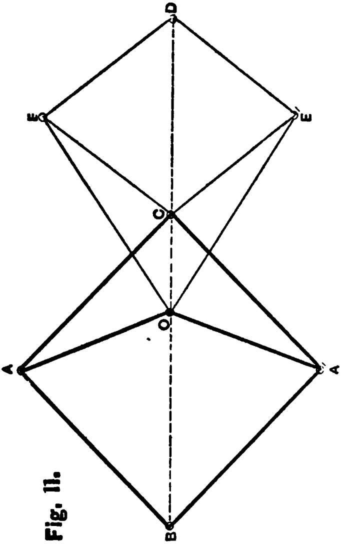


Fig. 11.

C'^2 become nothing, but their ratio is none the less definite and equal to $\frac{ED}{AB}$.

Extraction of square roots, or raising to the square, requires a complex combination of linkages which is only of a purely theoretical interest. This combination is made up of four simple Peaucellier elements and three of the *systeme varié* (Fig. 4) joined as shown in Fig. 12. In this combination OA gives the square root of OH, and inversely OH gives the square of OA.

In order to exhibit better the coupling of these elements we have represented separately the upper part of each element on a reduced scale (Fig. 13). The same letters have the same signification in the two figures.

The powers of the different elements are equal among themselves and have the following values, also given in Fig. 13,

$$(\tfrac{1}{2}C\sqrt{2})^2,\ C^2,\ C^2,\ C^2,\ (C\sqrt{2}),\ C^2 \text{ and } (\tfrac{1}{2}C\sqrt{2})^2$$

It follows from this by reason of equations (1) and (2) and taking $OA=x$ that we have consecutively:

$$OB=\sqrt{x^2+\tfrac{1}{2}C^2},\quad OC=\frac{C^2}{OB}=\frac{C^2}{\sqrt{x^2+\tfrac{1}{2}C^2}}$$

$$OD=\sqrt{\overline{OC}^2-C^2}=C\sqrt{\frac{C^2-2x^2}{C^2+2x^2}}$$

$$OE = ED - OD = \frac{C^2}{OD} - OD = \frac{C^3}{\sqrt{C^4 - 4x^4}},$$

$$OF = \frac{2C^2}{OE} = \frac{C\sqrt{C^4-4x^4}}{2x^2},\ OG = \sqrt{C^2 + OF^2}$$
$$= \frac{C^3}{2x^2}$$

and $OH = \frac{\frac{1}{2}C^2}{OG} = \frac{x^2}{C}$, then $\frac{OH}{C} = \left(\frac{x}{C}\right)^2$.

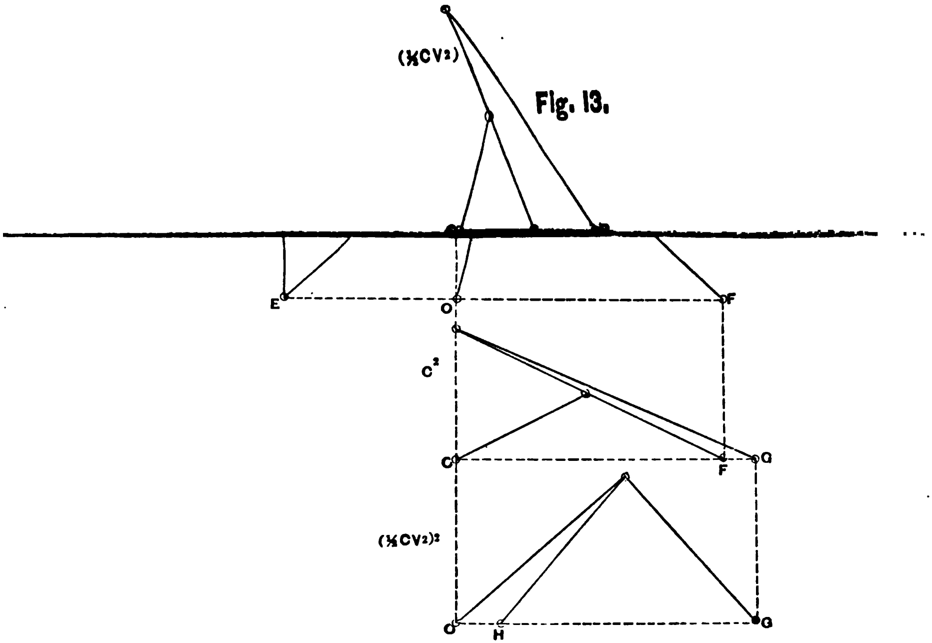
or taking C as the unit of the scale $OH = x^2$; OH represents then the square of OA, and inversely OA is the square root of OH. In Fig. 12 the unit C of the scale being 40 millimeters, $OA = 0^m5$ and consequently $OH = 0^m25$.

On the other hand, if OB (Fig. 13) represents the cosine of an angle *w*, it is clear from the preceding that OC, OD, OF, OG and OH represent the values respectively of

sec *w*, tan *w*, tan 2*w*, sec 2*w*, and ½ cos. 2*w*.

The system required for the extraction of the *cube root* is more simple, being a combination of three elements only, as shown in Figure 14.

Fig. 13.



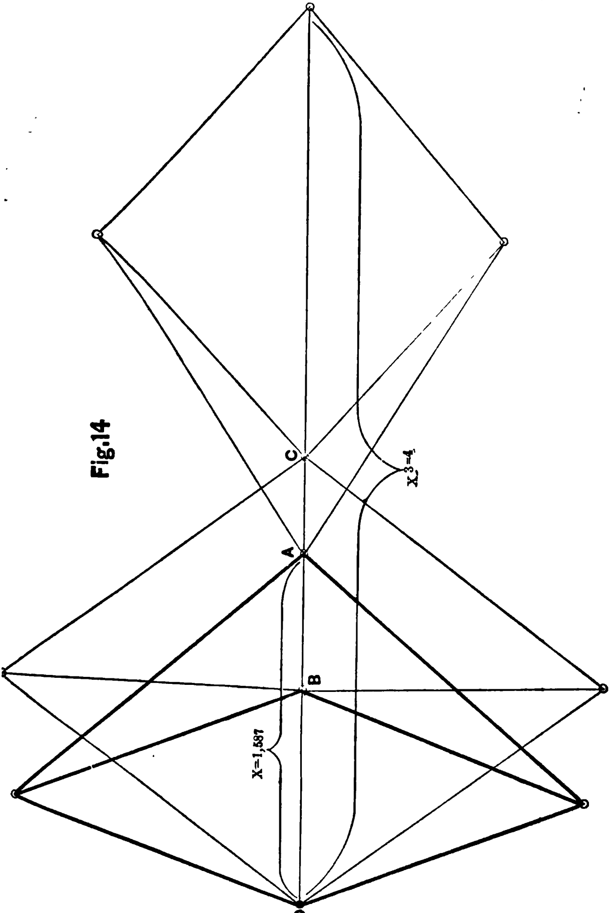


Fig.14

Since the elements have all the same power C^2 we have; taking $OA = x$;

$$AB=,\frac{C^2}{x} OB=OA-AB=\frac{x^2-C^2}{x}$$

$$BC=\frac{C^2}{OB}=\frac{C^2x}{x^2-C^2}$$

$$AC=BC-AB=\frac{C^2}{x(x^2-C^2)}$$

$$AD=\frac{C^2}{AC}=\frac{x(x^2-C^2)}{C^2} \qquad \text{and}$$

$$OD=OA+AD=\frac{x^2}{C^2}$$

then $\frac{OD}{C}=\left(\frac{x}{C}\right)^3$ or if $C=1$

$$OD=x^3.$$

The resolution of equations of any degree is of more importance than the extraction of roots, and the Peaucellier cell may be applied to these problems in different ways.

Thus, in some cases, the equation can be written in a complex form, the different parts of which may be calculated by means of this linkage.

Take for example the equation

$$4x^3-3x-a=0$$

which can be put under the form

$$a=x-\frac{4C^2}{\dfrac{C^2}{\dfrac{C^2}{x}-x}+\dfrac{C^2}{x}}$$

in which are presented three inversions;

$$\frac{C^2}{x},\quad \frac{C^2}{\dfrac{C^2}{x}-x}\quad \text{and}\quad \frac{4C^2}{\dfrac{C^2}{\dfrac{C^2}{x}-x}+\dfrac{C^2}{x}}$$

The solution then requires three elements whose powers are, C^2, C^2 and $(2C)^2$ which we will couple as indicated in Fig. 15.

Now taking OA $= x$, we have

$$AB=\frac{C^2}{x},\ OB=AB-OA=\frac{C^2-x^2}{x},$$

$$CB=\frac{C^2}{OB}=\frac{C^2x}{C^2-x^2},$$

$$AC=CB+AB=\frac{C^2}{x(C^2-x^2)},$$

$$AD=\frac{4C^2}{AC}=\frac{4x(C^2-x^2)}{C^2}$$

and

$$\mathrm{OD}=\mathrm{OA}-\mathrm{AD}=\frac{4x^3-3x\mathrm{C}^2}{\mathrm{C}^2}$$

then

$$\frac{\mathrm{OD}}{\mathrm{C}}=4\left(\frac{x}{\mathrm{C}}\right)^3-3\left(\frac{x}{\mathrm{C}}\right)$$

or expressing OD and x as functions of C;

$$4x^3-3x-\mathrm{OD}=0,$$

so that making OA $=x$ gives one of the roots of the equation, taking OD $=a$.

This system will also afford a solution of the famous problem, the *tri-section of an angle.* Since in the trigonometrical expression

$$\cos 3a=4\cos^3 a+3\cos a$$

we have $\frac{\mathrm{X}}{\mathrm{C}}=\cos a$, by taking $\frac{\mathrm{OD}}{\mathrm{C}}=\cos 3a$

Thus in Fig. 15 we have taken

$$\mathrm{OD}=20\cos 75^\circ=5.176$$

and we find $\cos a=\frac{\mathrm{OA}}{\mathrm{C}}\cdot\frac{18}{20}=0.9$

which gives for a a value of 25°.

Besides this solution there is another by means of curves, whose tracing is possible by the Peaucellier cell, and of

which we will speak further on. The trimetric curve serves especially for this purpose, and this curve is obtained equally well by the oval mechanism of Leonardo da Vinci.

In resolving the equation above we have written it under a more complex form, without concerning ourselves with the question of how far such a reduction is systematically possible, as such a question would be aside from the purpose of this paper, and furthermore, as another and more general method may be followed for the mechanical solution of equations of any degree.

This method which we believe to be new, is based upon the well-known property that in all complete equations arranged with reference to the powers of the unknown quantity, the co-efficient of the second term is equal to the sum of the roots taken with contrary sign; the co-efficient of the third term is equal to the sum of their products taken two and two; and the co-efficient of the fourth term is equal to the sum of the products three and three, taken with the opposite sign, etc., and finally, that the last term is equal to the product of all the roots, taken with the same or the contrary sign, according as the degree of the equation is even or odd.

The general form of an equation of the second degree would thus be:

$$x^2 - (a + b)\, x + ab = 0,$$

supposing for the sake of simplicity, that the roots a and b are real and positive.

Since in a negative Peaucellier element, of which the power $\overline{OB}^2 - \overline{BC}^2$ is ab, (Fig. 16) the product of the arms OB and BA remains in all possible positions of the system constant and equal to ab; it follows that the different values of OB and BA give all possible cases of the constant product ab, and of all these values it is necessary only that we take the two whose sum is $a + b$, we have only to make $OA = a + b$. This is easily accomplished by a divided scale along OA, and the roots are then immediately given by OB and BA.

For the different values of the product

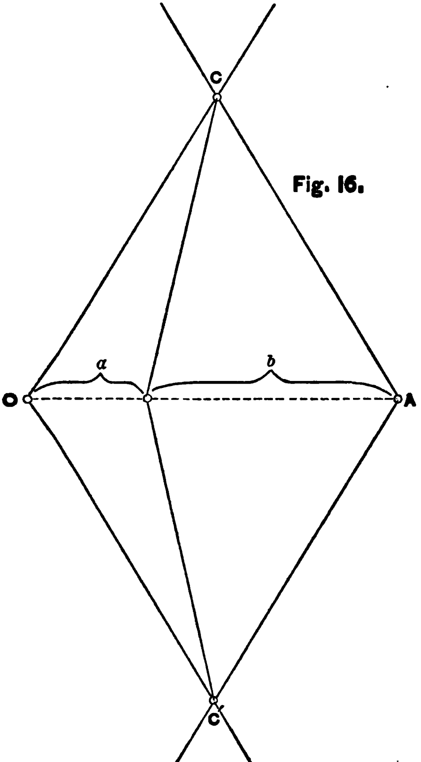


Fig. 16.

of the roots, the hinges C and C′ can be adjustable as in the proportional dividers, or quite as well, and what is more simple, the product of the roots can be reduced to the constant value $\overline{OC}^2 - \overline{CB}^2$ by multiplying by n, which changes the equation to

$$x_1^2 - n\,(a+b)\,x_1 + n_2\,ab = 0$$

in which n can be chosen so that

$$\overline{OC}^2 - \overline{BC}^2 = n^2\,ab.$$

For the case where the two roots are negative the signs may be changed, and if one only is positive or negative, the solution requires a positive Peaucellier element instead of a negative one.

In the mechanical solutions of equations of higher degrees it happens that some of the distances or lengths require to be transposed. The combination of links which effects this is designated the *transporter*, and is constructed as indicated in Fig. 17.

This arrangement is not the most simple; it can be replaced by the isosceles figure with crossed sides shown

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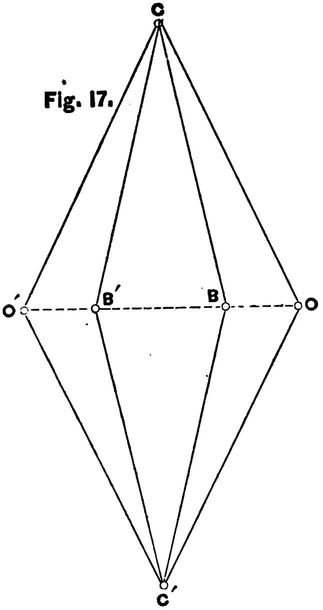
in Fig. 44. We prefer, however, to apply here the system of six rods as in Fig. 17 by reason of the symmetry.

Fig. 17.



In this arrangement the bars form two lozenges, one within the other, in such a manner that OB=O′B′, or in other terms, OB is always transported in O′B′.

The general equation of the third degree

$$x^3-(a+b+c)x^2+(ab+ac+bc)+x-abc=o$$

in which, for the sake of simplicity, we suppose the roots a, b and c, to be real and positive; is solved mechanically by aid of a transporter and three negative Peaucellier elements (Fig. 18) of which the powers $\overline{OF}^2-\overline{FB}^2$, $\overline{O'C}^2-\overline{BG}^2$ and $\overline{DH}^2-\overline{EH}^2$ are equal among themselves, and to abc or to n^3abc if we make the roots n times as great.

For this purpose it is only necessary to move A in the direction of O and A′ towards O′, so that OA shall be equal to $ac+bc+ab$ or to $n^2(ac+bc+ab)$ and so that $O'A'=a+b+c$ or to $n(a+b+c)$; the distances O′B, BC and CA′ giving then immediately the values of the roots directly, or these values multiplied by n.

The values are represented as follows:

$O'B=a$ $BC=b$ and $CA'=c$

$O'A'=a+b+c$

and $$\mathrm{OB}=\frac{\mathrm{OF}^2-\mathrm{FB}^2}{\mathrm{BC}}=\frac{abc}{b}=ac$$

$$\mathrm{BE}=\frac{abc}{a}=bc$$

since $$\mathrm{EA}=\frac{abc}{\mathrm{DE}}=\frac{abc}{\mathrm{CA}'}=ab$$

then $$\mathrm{OA}=ac+bc+ab$$

It follows inversely that O′B, BC and CA′ will represent the roots of the equation if O′A′ is equal to the coefficient of the 2nd term, and OA to that of the 3rd term.

It is understood that the connection of the elements is such that their arms remain constantly upon the line OA. In the present instance this condition is verified only for the two elements II′ and HH′ while the arm of the element GG′ coincides with a line passing through E, and the arm of the system FF′ with all lines passing through C; consequently the three points B, C and A form in general a triangle instead of being situated in a right line.

It is easy to obviate this by making

the points to move in a groove; then the lower half of the system may be dispensed with, and the working simplified. In every case it would be necessary to employ a divided scale with the groove, in order to be able to fix the points O, A, O′ and A′ at determined distances in order to read off the values of *a*, *b* and *c*.

For our present purpose, however, it is more in keeping to employ the articulated rods only.

It is clear that it will suffice to guide a single point such as C, in the direction passing through two other points B and E, which, by reason of the reciprocal connection of the four elements, are not forced to move in line with C; and consequently the conditions require that C be guided to move alone a line of variable length as BE.

This is easily accomplished in different ways: for example; by coupling the points B′, C and E′ of the two systems KLB′ and KLC′ (Fig. 18) respectively to the points B, C and E.

Each of the systems has eight links,

being formed of a *reciprocator* KPB′ of Peaucellier, and two rods of equal length KL and LP; but as they have the link KL common, the two systems form together a combination of fifteen links.

The elements of these compound systems are called *protracteurs* or *retracteurs*, and we shall see further on (in Figs. 24 and 25) that they enjoy the remarkable property that B′ and similarly C′ can move along the line B′E′ perpendicular to KL. This line passes invariably through the point E′ situated on KL or its prolongation; this being the geometrical place of B′ or C′ at the moment that PL or QL coincides with the prolongation of KL. So that although the distance of either B′ or C′ from E′ may change value, the two points are forced to follow the right line passing through E′.

We will not now consider the cases of imaginary roots; this subject requires separate treatment. But we have no doubt that the solutions of higher equations, by means of linkages, will be preferred to the graphical solutions of Lill, as explained in the work of Dr. Luigi Cremona. (Elements der Graphischen Calculs: Leipsig, 1875.)

Furthermore, we believe that the solutions explained above are capable of being to a considerable extent simplified, and that it is possible to discover analogous combinations for solutions of equations of higher orders.

The examples given will suffice to show how the Peaucellier cell may be used as an instrument for calculation.

It is true that some solutions require complex combinations, and so are only interesting from a theoretical point of view, but it is not impossible to bring about a simplification analogous to that which transforms the Peaucellier cell into a pantagraph.

It seems then that linkages promise to render some day excellent service as instruments for calculation, just as the Peaucellier cell has already proved serviceable as an instrument for drawing.

It is this latter property that we propose to study in the following chapter.

THE PEAUCELLIER CELL AS A DRAWING INSTRUMENT.

By placing the pole A of the positive system OCBC (Fig. 19) at the center D in such a manner as to cause it to describe an arc of a circle AA′, and maintaining the point O in a fixed position (a condition indicated in the diagram by the black circles at O and D) the point B will be found upon a curve BB′ whose equation is readily found.

Putting $\mathrm{OA}=v$, $\mathrm{OB}=w$,
$\mathrm{OB}'=z$, $\mathrm{AD}=\mathrm{A'D}=r$, and the angle

$$\mathrm{A'OD}=\varphi$$

We shall have

$$r^2=(v-r)^2+\overline{\mathrm{OA}}^{2\prime}-2(v-r)\mathrm{OA}'\cos.\varphi$$

whence

$$\mathrm{OA}'=(v-r)\cos\varphi\pm\sqrt{r^2-(v-r)^2\sin^2\varphi}$$

and since

$$\mathrm{C}^2=vw=\mathrm{OA}'z,$$

$$z=\frac{vw}{(v-r)\cos\varphi\pm\sqrt{[r^2-(v-r)^2\sin^2\varphi]}}=$$

$$=\frac{w}{v-2r}\left\{(v\text{-}r)\cos\varphi\mp\sqrt{[r^2\text{-}(v\text{-}r^2\sin^2\varphi]}\right\},$$

an expression both of whose signs are applicable.

It is the equation of a circle whose radius is R, and since for

$$\varphi=0,\ z=w\ \text{or}\ \frac{wv}{v-2r}$$

and for

$$\varphi=180,\ z=-\frac{wv}{v-2r}\ \text{or}\ -w$$

it is clear that

$$\mathrm{R}=\frac{1}{2}\left\{w+\left(-\frac{wv}{v-2r}\right)\right\}=\frac{1}{2}\left\{\frac{vw}{v-2r}+(-w)\right\}$$

then in general

$$\mathrm{R}=\mp\frac{wr}{v-2r}.$$

By transposing the origin of a quantity:

$\mathrm{OO}'=\mp\frac{wr}{v--2r}-w$ we shall have $\mathrm{O}'\mathrm{B}'=z'$

$=\sqrt{z^2+\overline{\mathrm{OO}'}^2+2z\mathrm{OO}'\cos\varphi}=$a constant$=$

$\mp\frac{wr}{v-2r}$

From this value of R we readily get

$$r=\frac{vR}{2R\mp w}$$

so that we can trace an arc of given radius R by employing two different radii.

The position of the circle however will not be the same for the two cases which result from the double value

$$OO'=\mp\frac{wr}{v-2r}-w$$

or again from the two values

$$-R-w \text{ and } +R-w.$$

In the negative system (Fig. 20) OA′, OB′ and also R preserve the same values as in the positive system, for the same values of v, w, r and φ.

The space occupied by the instrument is greater in case of (Fig. 20) than in that of (Fig. 19.)

The figure shows two arcs of circles, tangent to each other, having the same radius R traced by the two values of r (AD and AD′).

These circumferences approach each other as R increases and coincide when $v=2r$ or $R=\infty$.

For this value B describes a perpendicular to AB passing through B.

It is understood that the systems of figures 19 and 20 can assume a symmetrical position, that is to say, a position in which the center D is found upon the diagonal of the lozenge or upon its prolongation which passes through O.

But such position is not always possible, especially when r is large compared to v and w.

Nevertheless the formulas preserve their applicability, inasmuch as v and w are in all cases theoretically determinable.

Instruments for describing arcs of circles of great radius, are now found for sale under the name of compound compasses (*compas composé*) and although of relatively small dimensions will describe arcs of any radius.

Sylvester's Linkage described above affords equally a practical means of drawing arcs of large radius.

In Fig. 21 if the point O be fixed and the point A be made to describe a circle passing through O about the center D,

the point B will also describe the arc of a circle.

For by equation 3 we have, giving the letters the same signification:

$$OA.'OB'-\left(1-\frac{n'}{n}\right)OB'^2=\frac{n'}{n}C^2.$$

whence

$$OB'=y=-\frac{1}{2}\frac{nOA'}{n'-n}\pm$$
$$\sqrt{\left\{\frac{n'C^2}{n'-n}+\frac{1}{4}\frac{n^2OA'^2}{(n'--n)^2}\right\}}$$

and because

$$OA'=2A'D\cos(A'OD)=2r\cos\varphi,$$

$$y=-\frac{n}{n'-n}r\cos\varphi\pm$$
$$\sqrt{\left\{\frac{n'C^2}{n'-n}+\frac{n^2r^2\cos^2\varphi}{(n'-n)^2}\right\}}$$

This is the equation of a circle whose radius

$$R=\pm\frac{1}{n'-n}\sqrt{[n'(n'-n)C^2+n^2r^2]}.$$

Replacing C^2 by the value OB, the radical disappears, since

$$OA.OB-\left(1-\frac{n'}{n}\right)OB^2=\frac{n'}{n}C^2\text{and}\,OA=2r$$

then $$C^2=2r\frac{n}{n'}OB+\frac{n'-n}{n'}OB^2$$

and consequently

$$R=\pm\left(OB+\frac{nr}{n'-n}\right)$$

and also $$\frac{n'}{n}=1+\frac{r}{-OB\pm R}.$$

In this expression the two signs i cate simply curvature in opposite di tions. For $n=n'$ the radius beco infinite as in the case of the Peaucell instrument, and the arc BB′ become straight line perpendicular to OB at B

This remarkable property is n applied in mechanism for the conversi of alternate circular into perfect rec linear motion.

This device is an improvement upo Watt's "parallel motion," although mor complicated, because it is strictly exact

Instead of describing a perpendicular to the line OD, it is equally possible to trace a parallel to it. This, however, requires a more complex combination. It is made up of two ordinary elements and two of the *system varié* of Peaucellier, all of the same power, and united as shown in Fig. 22.

We have in this case

$$OH=\frac{C^2}{OA},$$

$$\overline{OI}^2=\overline{OH}^2-C^2=\frac{C^2(C^2-\overline{OA}^2)}{\overline{OA}^2}$$

$$OF=\frac{C^2}{OI}=\frac{C.\,OA}{\sqrt{C^2-\overline{OA}^2}}$$

$$\text{and } \overline{OB}^2=C^2+\overline{OF}^2$$

$$\text{or } OB=\sqrt{\frac{C^4}{C^2-\overline{OA}^2}}$$

Giving O a fixed position, and describing with A, a circle pssing through O having a radius $r=OD=AD$, and calling the angle $AOD=\varphi$, we shall have

$$OA=2r\cos\varphi$$

and consequently,

$$OB=\frac{C^2}{\sqrt{C^2-4r^2\cos^2\varphi}}$$

or making $C^2=4r^2$

$$OB=\frac{2r}{\sin\varphi}$$

Consequently the point B will describe a right line parallel to OD at a distance of $2r$ from it.

Somewhat more complicated again is the combination required for describing a right line making a given angle with the line through the fixed points.

This arrangement is shown in Fig. 23, and is composed of two of the transformed cells (*systeme varié*) and four ordinary Peaucellier elements, all of the same power C^2 with the exception of the positive elements LL′ and NN′, whose powers are equal respectively to C'^2 and C''^2.

We have in this case

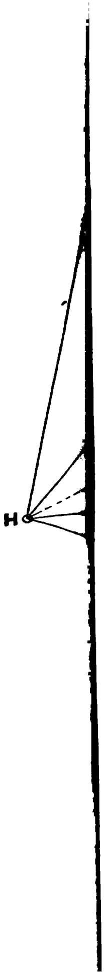
$$AM = \frac{C^2}{O}, \quad AP^2 = AM^2 - C^2 = \frac{C^2(C^2 - OA^2)}{OA^2}$$

$$AF = \frac{C^2}{AP} = \frac{C.OA}{\sqrt{C^2 - OA^2}},$$

$$AH^2 = C^2 + AF^2 = \frac{C^4}{C^2 - OA^2}$$

$$AK = \frac{C'^2}{AH} = \frac{C'^2}{C^2}\sqrt{C^2 - OA^2}$$

$$OK = OA - AK - OA - \frac{C'^2}{C^2}\sqrt{C^2 - OA^2}$$


H

$$OB=\frac{C''^2}{OK}=\frac{C''^2}{OA-\frac{C'^2}{C^2}\sqrt{C^2-OA^2}}$$

If, furthermore, A describes a circle which passes through O, we have

$$OA=2r\cos\varphi$$

and we take $C^2=4r^2$ and

$\frac{C'^2}{C^2}=\tan\alpha$ we obtain

$$OB=\frac{C''^2}{2r\cos\varphi-2r\tan\alpha\sin\varphi}$$

$$=\frac{C''^2\cos\alpha}{2r\cos(\alpha+\varphi)}$$

or $$OB\cos(\alpha+\varphi)=\frac{C''^2}{2r}\cos\alpha$$

Now tracing the line OQ making an angle α with OD, and dropping the perpendicular upon OQ from B, then

$$OQ=OB\cos(\alpha+\varphi)$$

and consequently $OQ=\frac{C''^2}{2r}\cos\alpha$

Then OQ having a constant value, the line described by the point B is a straight line perpendicular to OQ, and making an angle of $90-a$ with the line through the fixed points.

The distance OB′ is obtained immediately by $\varphi=0$ in the equation which gives OB. This distance then becomes $\frac{C''^2}{2r}$. In the figure C''^2 is taken 1431, $2r=40$ millimeters, and $a=40°$; the apparatus, therefore, describes two sides of a regular hexagon, of which the vertex B is at a distance OB′=36mm from O.

The particular case of a line passing through the fixed points, requires a combination much more complicated, and we believe should be omitted.

If in Figs. 19 and 20 the point O is united to D, we obtain respectively Figs. 24 and 25, of which the bars, eight in number, are movable each in relation to the others, so that any point describes a curve.

Thus the motion of C and B′ is circular relatively to the line which joins O and D; but the motion of any point between C and B′, as E for example, Fig. 26, describes a lemniscate.

The points O, C and C′ describe relatively to A′D arcs of circles, while B gives very remarkable curves which we will study for the particular case of OD=A′D. The point B′, as we have seen above, describes relatively to OD a right line passing always through the point B, Figs. 24 and 25. So that if the rod OD is prolonged and bent to a right angle at the point B, which is the geometrical position of B′ at the moment that A′D coincides with CD, the portion BB′ will pass constantly through B′, whatever the position of the system.

By this means we may lengthen or shorten the consecutive radii vectores of a curve, or in other words, we can derive from a given curve, a trace of the curve lengthened or shortened. This apparatus has therefore been called the *protracteur* or *retracteur*.

Bringing the point H and the two points K and K′ Fig. 27, (situated at a distance c from H, in the direction BB′) to the line OD, and describing with H a curve B′A, we shall have the other radii

Fig. 24.

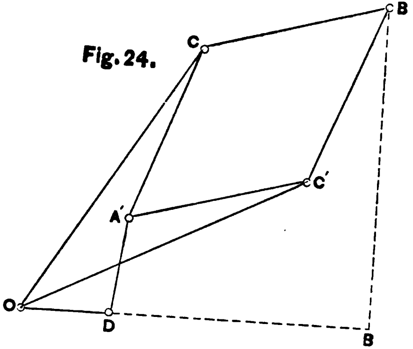
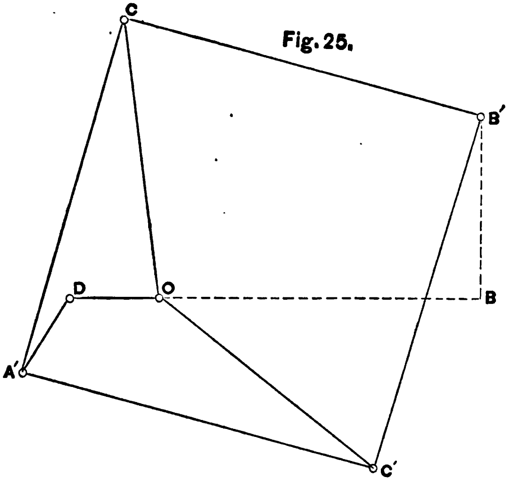


Fig. 25.



vectores, lengthened or shortened by the quantity c.

Thus for the particular case in which the curve is an arc of a circle passing through B′ and of which the center is at M, the curve shortened or lengthened is the *conchoid* with a circular base, known as the limaçon of Pascal.

Its equation is very simple: making the angle $HB'M=\varphi$, and the radius $B'M=HM=r$, we have

$$AK \text{ or } AK'=z=2r\cos\varphi\pm c.$$

For $c=2r$ the conchoid is transformed into the *cardioid*, and for $c=r$ it takes the name of *trimetric curve.* When the point H instead of following an arc of a circle describes a straight line as in Fig. 28, the joining of H to a second system causes the point K to describe the conchoid with a rectilinear base known as the conchoid of Nicomedes, and of which the equation is

$$B'K=z=\frac{B'B''}{\cos\varphi}\pm c.$$

The conchoid with a circular base has

been applied by Peaucellier to tr conic sections. Tracing for this pose, the radii vectores $BA=2r(\cos\varphi$ of the cardioid or conchoid w $HB=c=2r$ (which we have made in t case by means of a negative reciproc FF', we have

$$AE=\frac{C^2}{AB}$$

or by substituting for C^2 the two di tances EA and AB (for $\varphi=0$); that is t say, substituting the values E′A and AB as we have in the following, the equatio becomes

$$AE=\frac{AE.'AB'}{AB}=\frac{AE.'4r}{2r(1+\cos\varphi)}$$
$$=\frac{1}{\frac{1}{2AE'}(1+\cos\varphi)}$$

which is the polar equation of a parabola EE′ of which the parameter $p=4AE'$.

For $c=HB$, greater than $AH'=2r$ (Fig. 30) the equation $AE=\dfrac{AE'AB'}{AB}$

gives $$AE=\frac{2r+c}{2r.\cos.\varphi+c}.AE'$$

Fig. 27.

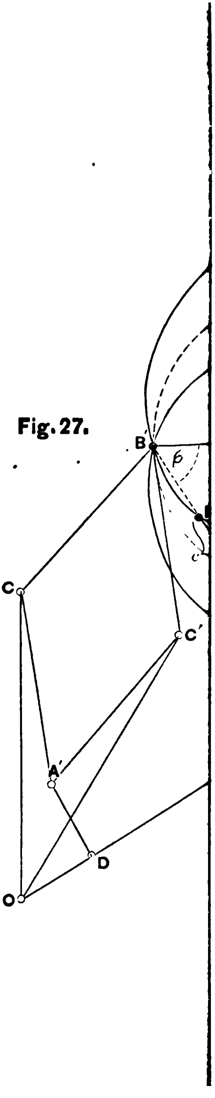
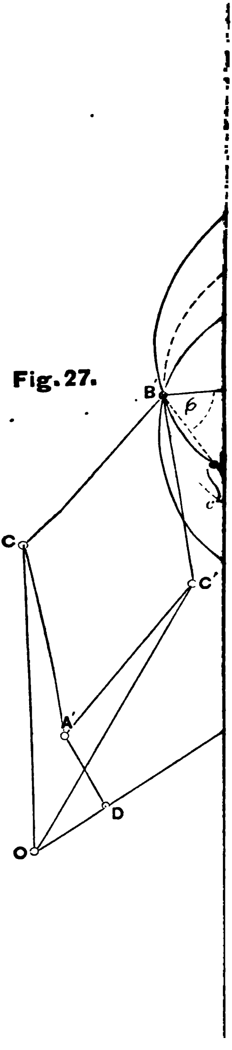
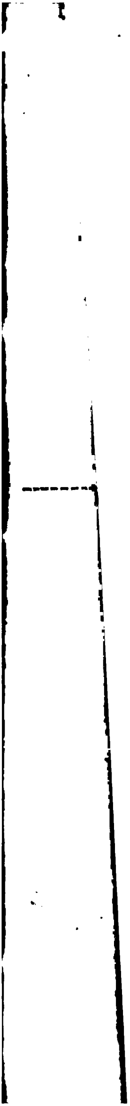


Fig. 27.





$$2b-2AE'\sqrt{\frac{2r+c}{2r-c}}=2AE'\sqrt{\frac{1}{AH'}}\cdot$$

while $$e=\frac{2rAE'}{2r-c'}$$

We will now examine the mov of B′ in relation to A′D (Figs. 24 a

Conformably with Fig. 32 in wh and B′ have been replaced by A the other letters retaining the sa nification, we shall have

$$C^2=OA(OA+OB)=O'B.'O'A$$

whence (by virtue of $OA=2r$ cos $O'A=2r$),

$$AB=\frac{O'B'}{\cos\varphi}-2r\cos\varphi.$$

The line FF′ having been drawn pendicular to AD at the distance A O′B′, we have

$$\frac{O'B}{\cos\varphi}=\frac{AF}{\cos\varphi}=AF'.$$

Furthermore $2r\cos\varphi=OA=AK$, from the supposition that the two cir OA and OK have the same radius, consequently $AB=AF'-AK=F'K$, f whence it follows that the motion of in relation to AD is along a *cissoid*.

When $O'B'=2r$, this becomes an ordinary cissoid or cissoid of Diocles, in which the asymptote is at the same time tangent to the circle. And this curve becomes a hypercissoid for $O'B'>2r$ and a hypocissoid for $O'B'<2r$.

In seeking for the inverse values of the consecutive radii vectores, we obtain in the first case (Fig. 33)

$$EA=\frac{C^2}{AB}=\frac{\cos\varphi}{\frac{2r\,(1-\cos^2\varphi)}{C^2}}$$

an equation of a parabola of which the parameter

$$p=\frac{C^2}{2r}$$

In the second case we obtain (Fig 34)

$$AE=\frac{AE.'AB'}{AB}=\frac{AE'(O'B'-2r)\cos\varphi}{O'B'-2r\cos^2\varphi}$$

or putting

$$AE'=2a \text{ and } O'B'=\frac{a^2}{a^2-b^2}\,2r$$

$$AE=\frac{\cos\varphi}{\frac{a}{2b^2}-\frac{a^2-b^2}{2ab^2}\cos^2\varphi}$$

Then the point E will describe an ellipse whose axes are

$$2a=\mathrm{AE}' \text{ and } 2b=2a\sqrt{\frac{\mathrm{O'B'}-2r}{\mathrm{O'B'}}}$$

$$=\mathrm{AE}'\sqrt{\frac{\mathrm{AB}'}{\mathrm{O'B'}}}$$

Finally in the third case where $\mathrm{O'B'}<2r$ we have (Fig. 35)

$$\mathrm{AB}=(\mathrm{AK}-\mathrm{AF})-\mathrm{OA}-\frac{\mathrm{A'F}}{\cos\varphi}$$

$$=\frac{2r\cos^2\varphi-\mathrm{O'B'}}{\cos\varphi}$$

and consequently the inverse radius vector

$$\mathrm{AE}=\frac{\mathrm{AB.'AE'}}{\mathrm{AB}}=\frac{(\mathrm{AO'}-\mathrm{O'B'})\mathrm{AE'}\cos\varphi}{2r\cos^2\varphi-\mathrm{O'B'}}$$

or putting

$$\mathrm{AE}'=2a \text{ and } \mathrm{O'B'}=\frac{a^2}{a^2+b^2}.2r$$

$$\text{then } \mathrm{AE}=-\frac{\cos\varphi}{\frac{a}{2b^2}+\frac{a^2+b^2}{2ab^2}\cos^2\varphi}$$

an equation of an hyperbola having the origin at the vertex, and of which the axes are

$$2a = AE' \text{ and}$$

$$2b = 2a\sqrt{\frac{2r - O'B'}{O'B'}} = AE'\sqrt{\frac{AB}{O'B'}}$$

The other movements of the system (Figs. 24 and 25) are all circular with the exception of that of the point D relative to CB′ or C′B′ which we will not discuss here.

In the arrangement of linkages, which we will proceed to describe, and which may be termed *conicographs*, the origin is at the same time at the fixed point and at the focus or summit of the curve.

For the ellipse and hyperbola the fixed point may be placed at the center and the construction becomes more elegant.

We will couple for this purpose two of the transformed Peaucellier Cells (éléments variés) OCLANC′ and BPNAL′P′ as is shown in Fig. 36.

The sides of the lozenges of the two elements are equal, and since the links AL and AL′ are constantly in prolongation, each of the other, they can be replaced by the single bar LL′, which will render AN superfluous.

This combination of eight links enjo the property, that the sum of the squar of OA and AB is constant.

Calling the connectors OC and PB spectively m and m', we have by virt of equation (2);

$$OA^2-AP^2=m^2-n^2 \qquad a$$
$$AB^2-AC'^2=m'^2-n'^2,$$

or adding the two equations;

$$DA^2+AB^2=m^2+m'^2-2n^2+AP^2+AC$$

but since AB is always perpendicular OA, we have;

$$AP^2+AC'^2=n^2,$$

and consequently;

$$OA^2+AB^2=m^2+m'^2-n^2.$$

Now, taking A as the fixed point, as dicated in Fig. 37, and making O mo in a circle which passes through A, t radius of which is $AD=r$, then the poi B describes a curve B′BR, which bea the name of *lemniscatoid*, and observi that;

$$AB^2=m^2+m'^2-n^2-OA^2$$

and $OA=2r\cos\varphi$,

it is clear that the equation of the curve is

$$\overline{AB}^2 = m^2 + m'^2 - n^2 - 4r^2\cos^2\varphi,$$

or by substitution for $m^2 + m'^2 - n^2$ of the value of AB when $\varphi = 0$

$$AB^2 = (AB'^2 - 4r^2) - 4r^2\cos^2\varphi,$$

while the substitution of the value of AB for $\varphi = 90°$ gives,

$$AB^2 = AR^2 - 4r\cos^2\varphi$$

When $AR^2 = 2r^2$ the lemniscatoid is transformed into the lemniscate of Bernouilli, and the equation becomes:

$$\begin{aligned} AB^2 &= -2r^2 + 4r^2\cos^2(90 - \varphi) \\ &= 2r^2\cos(180 - 2\varphi) \end{aligned}$$

and for $AR = 2r$ the curve becomes two tangent circles whose equation is

$$AB = 2r\sin\varphi.$$

Seeking the inverse value of the radius vector AB of the lemniscatoid, we shall have for the case of a real value of AB:

$$AE = \frac{C^2}{AB} = \frac{AB'.AE'}{\pm\sqrt{[AB'^2 - 4r^2 - 4r^2\cos^2\varphi]}}$$

or in taking

$$AE'=a \text{ and } AB''^2=\frac{4r^2b^2}{a^2-b^2}.$$

$$AE=\frac{1}{\pm\sqrt{\left\{\frac{1}{b^2}-\frac{a^2-b^2}{a^2b^2}\cos^2\varphi\right\}}}$$

consequently the curve EE′ described by the point E is an ellipse whose axes are:

$$2a=2AE' \text{ and } 2b=\frac{2AE'.\,AB'}{\sqrt{4r^2+AB'^2}}$$

On the other hand, if AB′ is imaginary, we have, Fig. 38:

$$AE=\frac{C^2}{AB}=\frac{AR\,.\,AE}{\pm\sqrt{[AR^2-4r^2\cos^2\varphi]}}$$

or making

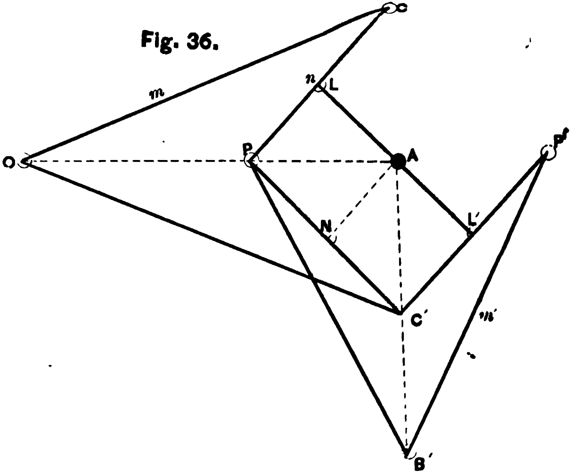
$$AE'=a \text{ and } AR=\frac{4r^2b^2}{a^2+b^2}$$

$$AE=\frac{1}{\pm\sqrt{\left\{\frac{1^2}{a^2}-\frac{a^2+b^2}{a^2b^2}\cos^2\varphi\right\}}}$$

and consequently the curve EE′ described by the point E is a hyperbola whose axes are:

$$2a=2AE' \text{ and } 2b=\frac{2AE'.\,AR}{\sqrt{4r^2-AR^2}}.$$

Fig. 36.



This tracing of the ellipse and hyperbola is due to Sylvester, and is especially remarkable for the simplicity of the transformation of cos φ to sin φ.

To appreciate this, it is necessary to recall the fact that it required in the case of Fig. 23, a combination of five elements, or thirty links, in order to pass from cos φ to sin φ, and the same combination was required to change

$$OA \text{ to } AK=\frac{C'^2}{C^2}\sqrt{C^2-OA^2}$$

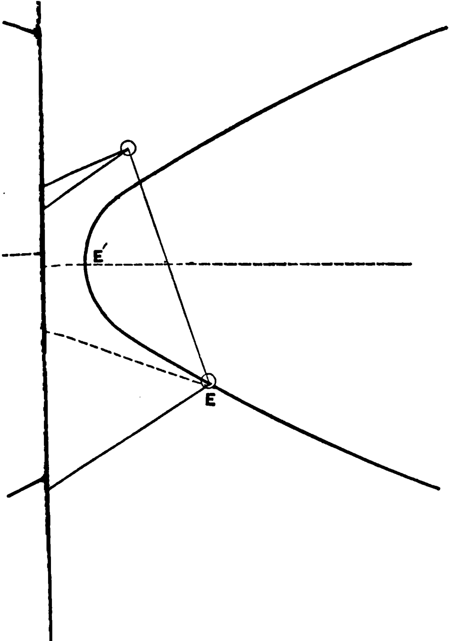
while the same operation, that is to say, the change (Fig. 36) from

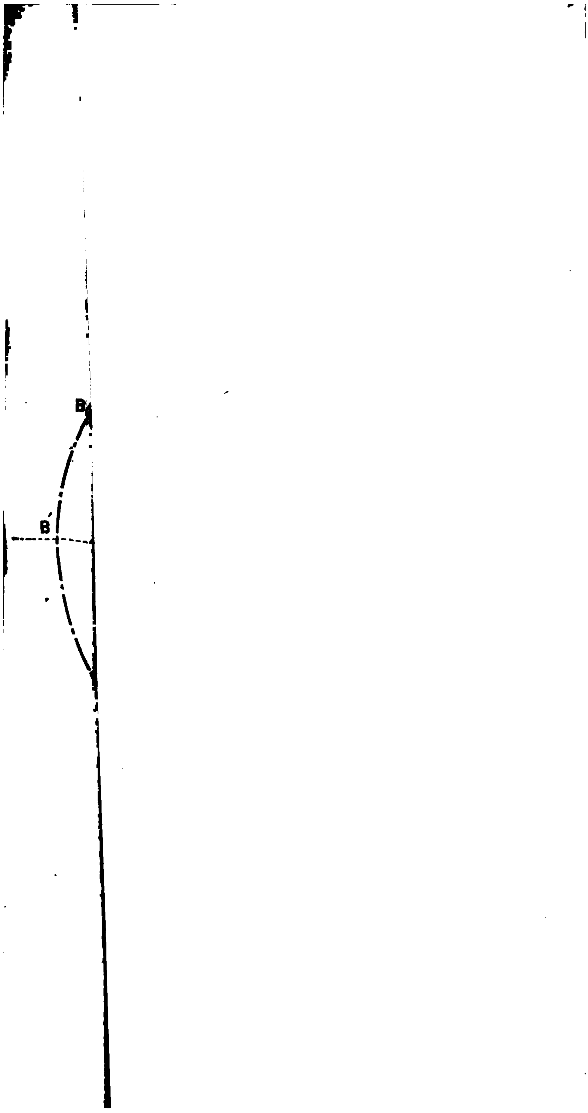
$$OA \text{ to } AB=\sqrt{C''^2-OA^2}$$

is here effected by a single element of eight bars.

In the first case, AK falls in the direction of OA, while in the second AB is perpendicular to OA.

But this system of eight links of Sylvester can be still further simplified, for the transformed cell of Peaucellier, containing only six bars, suffices to trace the conic sections around their center.


E'
E


B
B'

Uniting for this purpose, as shown in Fig. 39 and 40, the *element variè* CC′ to the ordinary element GG′, and describing from O a circumference which passes through the fixed point A, and of which the radius $AD=OD=r$, we shall have in this case, by virtue of eq. 2,

$$AB^2=C^2+OA^2$$

or supposing that B and O come to B′ and O′ when OD takes the direction AD,

$$AB'^2=C^2+O'A^2,$$

and further,

$O'A=2r$ and $OA=2r.\cos O'AO=2r\cos\varphi$

then $\quad AB^2=AB'^2 -4r^2+4r^2\cos^2\varphi$

or replacing φ by its complement φ'

$$AB^2=AB'^2-4r^2\cos^2\varphi'.$$

Consequently taking AB′, Figs. 39 and 40, equal to AR of Figs. 37 and 38, the curve described by the point B will be the same *lemniscatoid* in both cases.

The inverse value of AB is

$$AE=\frac{1}{\pm\sqrt{\frac{AB'^2-4r^2}{AE'^2.AB'^2}+\frac{4r^2}{AE'^2-AB'^2}\cos^2\varphi}}$$

and gives consequently a conic se
and for $AB' > 2r$ an ellipse (Fig.
whose axes are

$$2a = \frac{2AB'.AE'}{\sqrt{AB'^2 - 4r^2}}$$

and $$2b = 2AE',$$

and for $AB < 2r$ an hyperbola (Fig.
having for axes

$$2a = 2AE'$$

and $$2b = \frac{2AE'.AB'}{\sqrt{4r^2 - AB'^2}}$$

This conicograph, the fourth which
have made known, improves not onl
simplicity upon the third, but posses
the further advantage of describing w
great facility the more flattened port
of the ellipse.

A conicograph more simple still
based upon the property that the cu
described by ε, Fig. 26, under determi
nate conditions is an inverse conic.

This curve is described by means
three links, by coupling the point B'
the center O' of the circular motion; b

as the deduction of the general formula requires very complex calculations, and as we have restricted this note mostly to the systems of six links, we will omit further reference to it.

Meanwhile we will present the calculation for the more important case in which $OC=OO'$, $CB'-B'O'$ and the point E is found on CB'.

Drawing in Fig. 41, where B' has been replaced by B, the right line EN parallel to OC, EO'' parallel to the diagonal OB, and from the point O'' the intersection of EO'' and OO' a right line $O''M$ parallel to $O'B$; it is evident that the triangle is equal and similar to the triangle MOO'', and consequently,

$$NE=OO''.$$

And since, furthermore,

$NE : OC : : BE : BC$ and OC, BE and BC, which we will designate respectively by p, w and q are of invariable length, it is necessary that

$$OO''=NE=\frac{pw}{q}.$$

The length OO″ is therefore equally invariable, and we may choose O″ as the origin of a system of polar co-ordinates.

Taking O″E$=z$; angle CBO$=$OBO′ $=\beta$; angle COB$=$BOO′$=$EO″O$=\varphi$; angle XCY$=2\varphi$, and XCZ$=\beta-\varphi$, and since

$$\mathrm{VE}=\mathrm{BU}+\mathrm{WB}$$

$$z\sin\varphi=q\sin(\beta+\varphi)+w\sin(\beta-\varphi)$$

$$=q\sin\beta\cos\varphi$$

$$+q\sin\varphi\cos\beta+w\sin\beta\cos\varphi-w\sin\varphi\cos\beta$$

$$=(q+w)\sin\beta\cos\varphi+(q-w)\sin\varphi\cos\beta.$$

Furthermore we have in the same figure

$$p.\sin\varphi=q\sin\beta$$

or $$\sin\beta=\frac{p}{q}\sin\varphi, \text{ and}$$

$$\cos\beta=\pm\sqrt{1=\frac{p^2}{q^2}\sin^2\varphi}$$

so that the value of z becomes after substitution of $\sin\beta$ and $\cos\beta$;

$$z=\frac{p}{q}(q+w)\cos\varphi\pm$$

$$\sqrt{(q-w)^2-\frac{p^2}{q^2}(q-w)^2\sin^2\varphi}$$

an equation of which both signs may be satisfied.

The curve ET(Fig. 41) which this figure represents, can be obtained by a point on a curve, which rolls upon another curve, in the same manner as a point in a circumference which rolls on a line generates the cycloid.

M. Mannheim has made a study of these curves to which Reuleaux has given the general names of *polar curves* (*courbes ou lignes polaire*) (*polbahnen*).

He has found that the displacement of CB relative to OO″, and consequently the curve described by E is produced by an oval of Descartes rolling upon an oval of Descartes.

In seeking for the inverse value of z or O″E by the reciprocater DD′(Fig. 42) we obtain for the equation of the curve described by F:

$$O''F = \frac{C^2}{z} = \frac{1}{\frac{p}{q}\frac{(w+q)}{C^2}\cos\varphi \pm \sqrt{\left\{\frac{(q-w)^2}{C^4} - \frac{(q-w)^2}{C^4}\frac{p^2}{q^2}\sin^2\varphi\right\}}}$$

But the ellipse and the hyperbola of which the axes were a and b have an equation of the same form, supposing that the origin is situated upon the prolongation of the principal axis at a distance O″G$=v-a$ (Fig. 42) beyond the vertex, or at a distance v from the center. In effect, in this case, the general equation becomes:

$$z' = \frac{1}{\frac{v}{v^2-a^2}\cos\varphi \pm \sqrt{\left\{\frac{a^2}{(v^2-a^2)^2} - \frac{a^2[b^2 \pm (v^2-a^2)]}{b^2(v^2-a^2)^2}\sin^2\varphi\right\}}}$$

The positive sign of the radical relates to the ellipse, and the negative sign to the hyperbola, while the parabola whose parameter is b, and whose vertex is situated at a distance O″G$=v$ from the origin O″ has for an equation,

$$z_1 = \frac{1}{\frac{1}{2v}\cos\varphi \pm \left\{\sqrt{\frac{1}{4v^2} - \frac{l+4v}{4lb^2}\sin^2\varphi}\right\}}$$

It follows that the point F describes

a conic section FB and that consequently the system of links devised by Roberts is a conicograph having three fixed points.

By equating the co-efficients, we find for the ellipse and the hyperbola

$$\frac{(w+q)}{C^2}\cdot\frac{p}{q}=\frac{v}{v^2-a^2},\qquad \frac{(q-w)}{C^2}=\frac{a}{v^2-a^2}$$

and $$\frac{(q-w)^2}{C^4}\cdot\frac{p^2}{q^2}=\frac{a^2[b^2\pm(v^2-a^2)]}{b^2(v^2-a^2)};$$

for the parabola $$\frac{1}{2v}=\frac{(w+q)p}{C^2q},\quad \frac{1}{2v}=\frac{q-w}{C^2}$$

and $$\frac{l+4v}{4lv^2}=\frac{(q-w)^2}{C^4}\cdot\frac{p^2}{q^2}.$$

The first case gives rise to three equations with four unknown quantities a, b, v. and C, of which one must consequently be chosen arbitrarily.

In supposing v known the axes are:

$$2a=\frac{2qv(q-w)}{p(w+q)}$$

and

$$2b=2qv\sqrt{\pm\frac{p^2(q+w)^2-q^2(q-w)^2}{p^2-q^2}},$$

and the power

$$C^2=\frac{v(q+w)[p^2(q+w)^2-q^2(q-w)^2]}{pq(q+w)^2};$$

If, on the contrary, the power is given, we have

$$2a=\frac{2(q-w)q^2C^2}{p^2(q+w^2)-q^2(q-w)^2},$$

$$2b=\frac{2q^2C^2}{\sqrt{\pm(p^2-q^2)[p^2(q+w)^2-q^2(q-w)^2]}},$$

and the distance $O''G+a$ from the origin at the center of the ellipse or hyperbola is

$$v=\frac{pq(w+q)C^2}{p^2(q+w)^2-q^2(q-w)^2};$$

the positive sign belongs to the ellipse, and the negative to the hyperbola.

In the second case we have

$$w=\frac{q(q-p)}{p+q},$$

$$v=\frac{l(p^2-q^2)}{4q^2} \text{ and } C^2=l(p-q)\frac{p}{q}$$

or the parameter

$$l=\frac{C^2q}{p(p-q)}.$$

In figure 42, $p=6$ and $q=3$ centimeter, while $v=2$ centimeters, which renders v negative and $=-1$ centimeter, and gives for the parameter a length of $2\frac{2}{3}$ centimeters; in which case $C^2=16$.

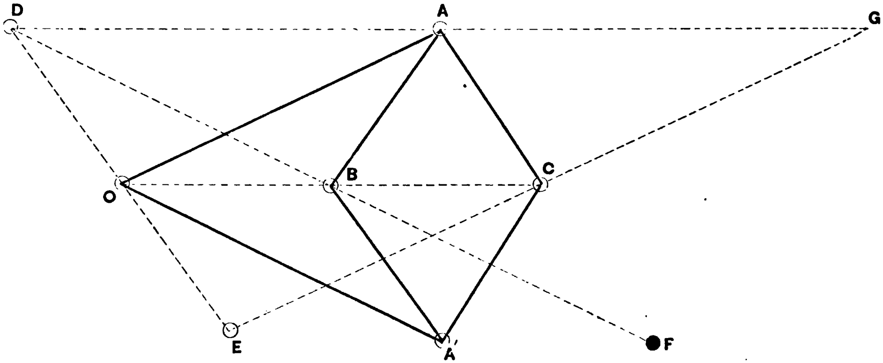
Then it results from the values of v that in the conicograph of Roberts, the origin O'' can never be either at the vertex or center of the curve.

Nevertheless this conicograph is to be preferred to the four preceding ones, especially because of the small number of links required, and which may even be still further reduced to seven.

The same simplification which was applied to the pantograph of six bars, Fig. 7, to produce the pantograph of four (Fig. 8) can equally be applied to the *reciprocator*, although not in so ready a manner.

Through the point C' of the positive cell AA' (Fig. 43) draw a right line CE parallel to OA, and from the point O a line OE parallel to AC; then prolonging EC and EO so that $EG=2EC$, and $ED=2OE$, we shall have the points D, A and G in the same right line, (just as in

Fig. 43.



the pantograph Fig. 8 the points O, E′ and B fell), and also AD=AG.

From the point D draw the right line DF passing through B, and make DF=2DB; since DO is equal and parallel to BA′, DB must be equal to OA′; that is to say, it preserves an invariable length.

The same may be said of CE and OE, so that CA′, A′B and A′O can, without modifying the movement of B, be replaced by the links CE, ED and OD.

On the other hand the triangle DBA is similar to the triangle DFG, and consequently FG=2AB; also of invariable length, so that the three remaining bars of the losenge OA, AB and AC can be suppressed by uniting F to G.

This leads us to the combination shown in figure 44, having three fixed points O, B and C situated on a right line parallel to the diagonals DG and EF, while the four links are in pairs of equal length.

The relation between the arms OB and OC is deduced immediately from Figs. 43 and 44 which give

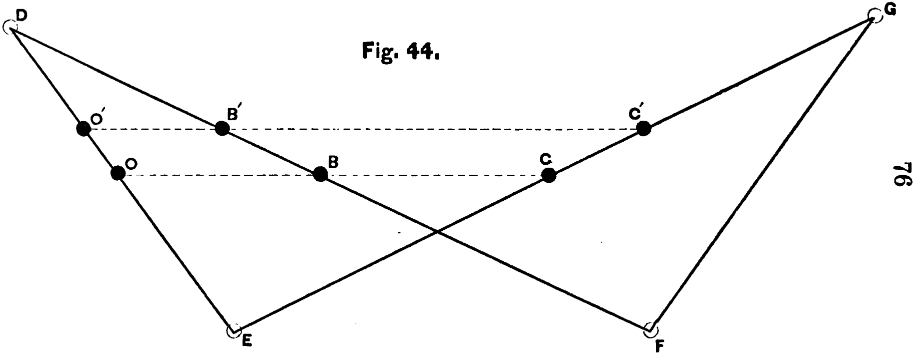


Fig. 44.

$$OB\,.\,OC = OA'^2 - AC^2 = BF^2 - DO^2$$
$$= \tfrac{1}{4}(DF^2 - DE^2)$$

while for the arbitrary arms O′B′ and O′C′ we have because

$$DO' : \tfrac{1}{2}\,DE :: O'B' : OB$$

and

$$EO' : \tfrac{1}{2}\,DE :: O'C' : OC$$

therefore

$$O'B'.\,OC' = 4OB\,.\,OC.\,\frac{DO'.\,EO'}{DE^2}$$
$$= \frac{DO'.\,EO'}{DE^2}\,(DF^2 - DE^2)$$

It was Hart who first noticed this property, and gave a direct demonstration of it.

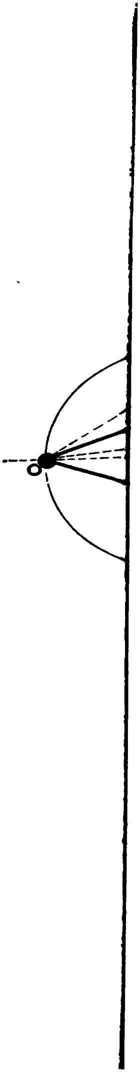
We may then, in the preceding constructions, replace the reciprocator of Peaucellier by that of Hart, which we will describe, and which affords the transformation from alternating circular to alternating rectilinear motion, by means of five links instead of seven.

There is, however, no great practical utility in this property, for if the number of links is diminished by two, the number of independent attached points is increased by two whenever the apparatus is employed in combination with others.

Moreover, any point situated on ED or fixed to ED as is O′ in Fig, 44, describes relatively to FG (or EG) an inverse conic section, so that this element, like that of Roberts, can serve as a conicograph.

The particular movement of this system (or if preferred of the two elements of Hart) can easily be derived from the elements of Roberts. We find that in the element of Hart, the motion depending upon the crossing or non-crossing of the sides of the isosceles trapezeum, is like that of the Roberts element under the conditions of $q>p$ or $q<p$ respectively.

In this case, however, the polar lines of the curve O′ are no longer ovals of Descartes, but equal ellipses, whose major axes coincide with ED and FG and are equal to DF and EG, while the foci are at D and E for one and at F and


O

G for the other. Or, again, the two polar curves may be two equal hyperbolas of which the larger axes DF and EG coincide, and of which the foci are at D and F, or at E and G.

To these examples might be added a great many others, but we believe these to be sufficient to prove the great practical utility of the different systems of articulated linkages.

Until recently the number of curves that could be traced mechanically was quite limited; among the conic sections the ellipse alone was described by the elliptical compass, and by the apparatus of Leonardo da Vinci; leaving out the trace obtained by the tense thread.

The two methods referred to are based on the property—that all points of a straight line of determinate length describe ellipses when the extremities of the line are made to slide along the lines bounding a right angle. These devices have been extensively used in practical draughting, and have been regarded as indispensable.

Da Vinci's apparatus is described in many works; among others in Hartmann's *Metalldreherei;* in Thon's *Drehkunst*, and in the more recent work on Kinematics by Reuleaux.

The oval obtained by the universal lathe of Kochand Muller, in which the friction, and consequently the wear, is much less than in Da Vinci's apparatus, is not quite symmetrical, and is therefore not an ellipse.

The linkages for describing the other conic sections will find, without doubt, applications as extensive as those above stated. The parabola, for example, is the proper form for a reflector, and for the outline of solids of equal resistance throughout. Furthermore, the parabola may, from an æsthetic point of view, profitably replace the arc of the circle in our machines and other constructions, as is urged by Reuleaux.

The systems of linkages are, in our opinion, of great value, both from the theoretical and practical point of view.

The following memoirs have already

appeared upon this subject: *Des systemes articulés simple et multiples, et de leur applications*, by Saint Loup, and, *Sur les systemes de tiges articulés*, by Liguine.

We may remark here that Liguine, who departs from the plan of the system of Fig. 9, urges that it is not indispensable, that the bars should be in pairs of equal length, in order that the points O, B and C remain in a right line. He says that it will suffice if the diagonals AA′ and II′ are perpendiculars upon OC(Fig. 50); which is an immediate consequence of the fact that the formulas contain, not the absolute values of the sides, but the difference of their squares, as has been already mentioned.

But this restriction of Liguine is too particular, since in any quadrilateral OBCD (Fig. 46), whose diagonals may intersect at any angle, a point A of a diagonal can always be taken so related to two of the sides that O, A and B shall be in a line, and so that the product of OA and BA shall be constant. We pro-

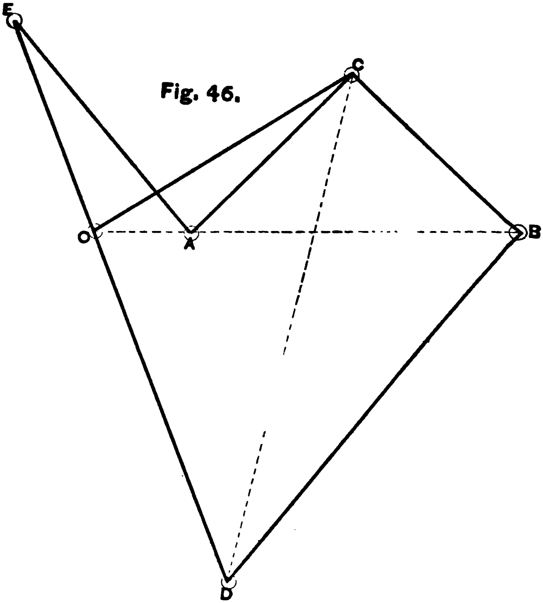


Fig. 46.

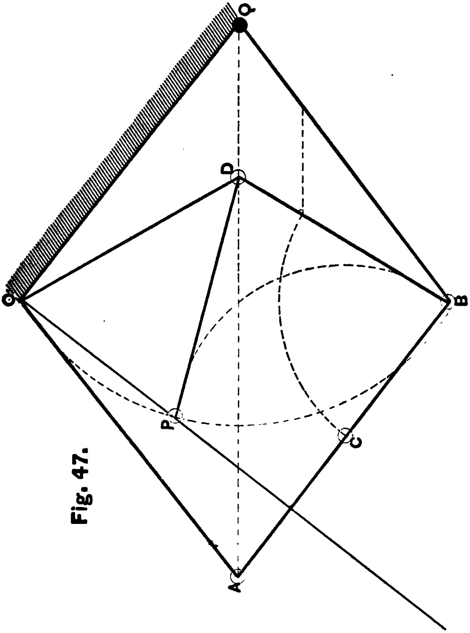


Fig. 47.

pose to demonstrate this case in a future note, in which we will give special attention to this new element, and to some other systems, such as that of Kemp, represented in Fig. 47, and in which P describes a perpendicular PO to the line OB at O; which results whenever

$$\mathrm{OA=AB=BQ=QO}$$

$$\mathrm{OD=DP=DB}$$

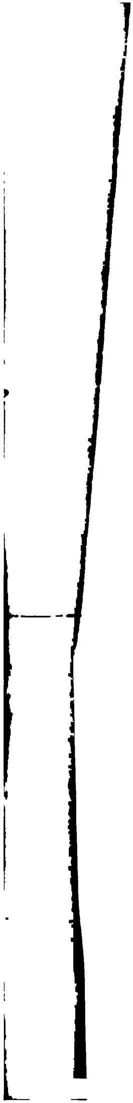
and

$$\mathrm{PC=BC=\frac{OD^2}{OQ}}$$

[NOTE BY THE FRENCH TRANSLATOR.]

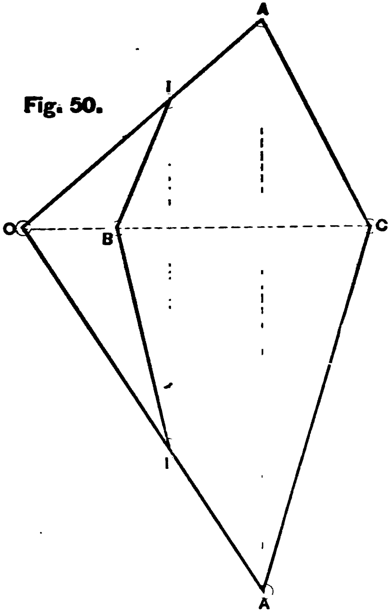
The interest attaching to the beautiful invention of Peaucellier is incontestable, either from the kinematic or mathematical point of view. But we may nevertheless be permitted to remark that the complexity of some of the solutions neutralizes their beauty, and renders doubtful their utility.

In mechanism simplicity is of the first importance, and ought always to be one of the principal factors of a new inven-



tion. For instance, in one example we have seen that the extraction of the square root by means of linkages re-

Fig. 50.



quires the combination of three *elêments variés*, and four ordinary Peaucellier elements; altogether forty-two links or bars.

It is very probable, as the author remarks, that this combination may be much simplified. (Since a parabola can be traced by a conicograph relatively simple, we may employ the latter for extracting the square root by making the parameter equal to unity, for then $y^2=x$ and $y=\sqrt{x}$).

But we do not wish to criticise here this particular case, but we may call attention to the dangers of the tendency of mathematicians to enjoy such difficulties; a tendency for which the complex combinations of the Peaucellier cell affords great opportunities for development.

One of the most direct applications of this cell is its substitution for the parallelogram of Watt. Figs. 48 and 49 exhibit the mode of application to the piston-rod of a steam-engine. In Fig. 48 the fixed point O coincides with the center of motion of the beam, and D is a fixed point independent of the beam. B

is fastened to the piston-rod. The walking-beam is the seventh bar OC. This is a positive element.

A better application is shown in Fig. 49, since the losenge itself forms the walking-beam, having O for a center of motion, and D as the independent fixed point.

The combinations described by the author for the mechanical solution of equations of the higher degrees are, without doubt, very ingenious, and we feel convinced that if some one would make them for sale, many persons would find them of great use.

Referring to the generalization that the author makes of the problem of rectilinear movement, by means of linkages, and which leads to the new element of Fig. 46, we propose for it the name of the "Element of De Roos" in honor of the inventor.

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